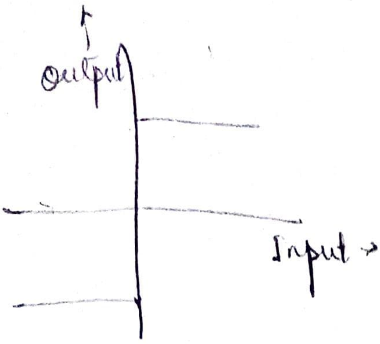
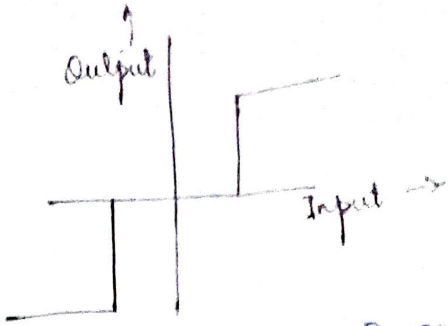
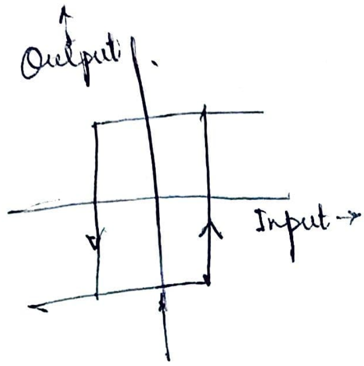


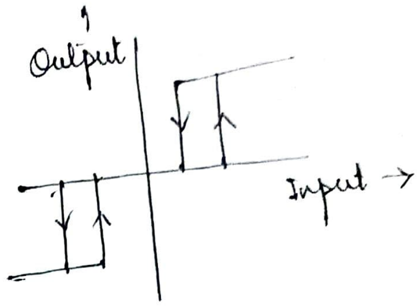
(a) Ideal Relay



(b) Relay with dead zone



(c) Relay with pure hysterisis



(d) Relay with dead zone & hysterisis

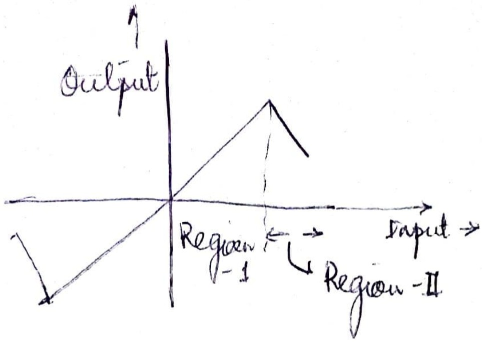
INVESTIGATION OF NON-LINEAR SYSTEMS:

The non-linear systems can be investigated by a number of tools.

Linearization:

If the operation of the linear system is in a restricted range about the operating point, the non-linear system can be approximated by a linear model.

But many physical systems are decidedly non-linear. The advantages of linear theory can be extended to such cases by piecewise linearization. In such a case, the response obeys a certain linear differential equation in one region of operation and a different one in another region. Depending upon the value of the region parameter, we switch from one differential equation to the other, such that the end conditions of the first are carried over as initial conditions of the other. Such systems are called piecewise linear systems.



Piecewise Linear Characteristic

In more complicated situations, where the system is distinctly non-linear, piecewise linearization is cumbersome and time-consuming. For such cases, two methods of analysis are available, which have an appreciable degree of generality.

(i) Phase plane method

(ii) Describing Function method.

Phase - Plane Method:

This is basically a graphical method, from which information about transient behaviour and stability is easily obtained by constructing phase trajectories.

From the point of view of practical utility, this method is restricted to 2nd order systems, excited by step or ramp inputs.

Describing Function Method:

This method is based on harmonic linearization. We assume here that the input to the non-linear component is sinusoidal and depending upon the filtering properties of the linear part of the overall system, the output is adequately represented by the fundamental frequency term in the Fourier Series.

This approximation is not restricted to small signal dynamics.

It usually gives sufficiently accurate information about system stability and limit cycles, but does not give a trustworthy information about transient response.

Phase-Plane Method:

Consider an unforced linear spring-mass-damper system, whose dynamics is described by

$$M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = 0.$$

Let this system be activated by initial conditions only.

The above equation can be written in standard form as

$$\frac{d^2x}{dt^2} + 2\xi\omega_n\frac{dx}{dt} + \omega_n^2 x = 0.$$

This is the equation of a 2nd order system and let the state variables be

$$x_1 = x$$
$$x_2 = dx/dt = \dot{x}$$

The state variables defined in this fashion are called phase variables.

We can write,

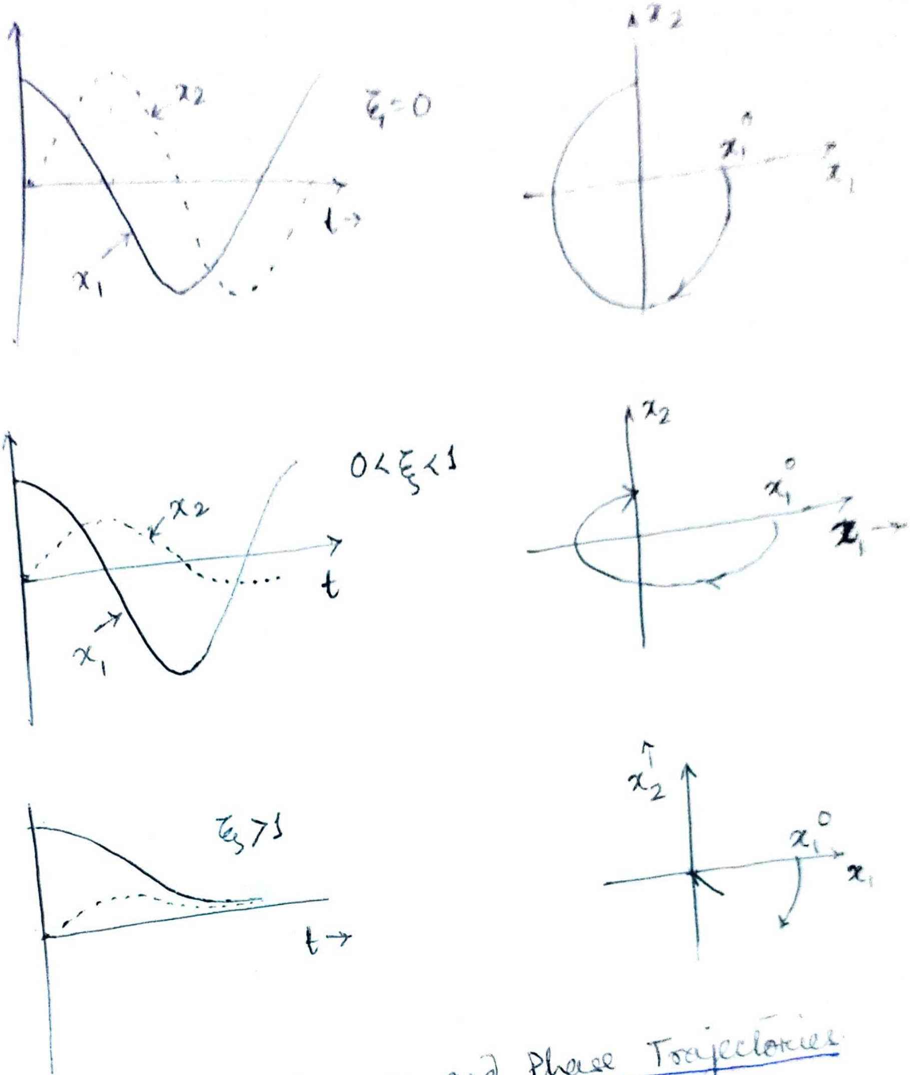
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{d^2x}{dt^2} = -\omega_n^2 x_1 - 2\xi\omega_n x_2.$$

When the differential equations describing the dynamics of the system are non-linear, it is in general not possible to obtain a closed-form solution of x_1, x_2.

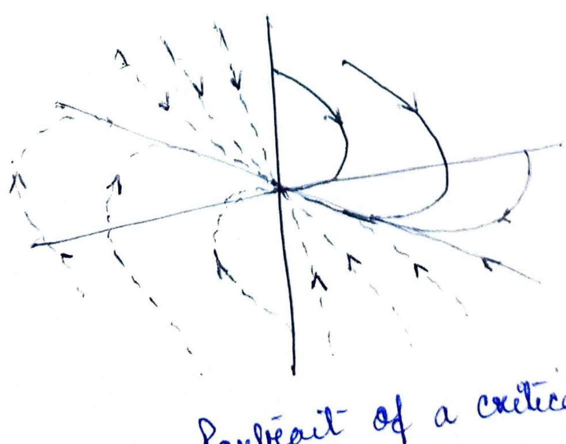
(For example, when the spring force is $K_1x + K_2x^3$)

In such situations, a graphical method, known as the phase-plane method is found to be useful.

The coordinate plane with axes that correspond to the dependent variables $x_1 = x$ and its first derivative $x_2 = \dot{x}$ is called the phase plane. The curve described by the state point (x_1, x_2) is called the phase trajectory. A family of phase trajectories corresponding to different initial conditions are called phase portraits. In a phase portrait, larger or smaller initial conditions produce geometrically similar trajectories.



Time Response Curves and Phase Trajectories



Phase Portrait of a critically-damped system.

SINGULAR POINTS:

Consider a general time-invariant system described by the state equation

$$\dot{X} = f(X, U)$$

If the input vector U is constant, it is possible to write the above equation in the form

$$\dot{X} = f(X).$$

A system represented by an equation of this form is called an autonomous system. For such a system, the points in the phase-space at which derivatives of all the state variables are zero, are called singular points. These are also the equilibrium points. If the system is placed at such a point, it will continue to lie there if left undisturbed (the derivatives of all the phase variables being zero, the system state remains unchanged.

Consider the linear or linearized system

$$\dot{X} = AX.$$

$X_e = 0$ is the only solution of the equation $\dot{X} = AX_e = 0$ if the determinant A is non-zero. Equivalently, if all the eigen values of the system are non-zero, then the origin is the only singular point.

In general, if X_e is the equilibrium point, it is convenient to shift the origin of the coordinate plane to X_e, with the transformation $\tilde{X} = X - X_e$.

The examination of the phase-plane trajectory can best be done in the canonical form of representation. The system can be transformed to diagonal canonical form by similarity transformation.

Using the linear transformation,

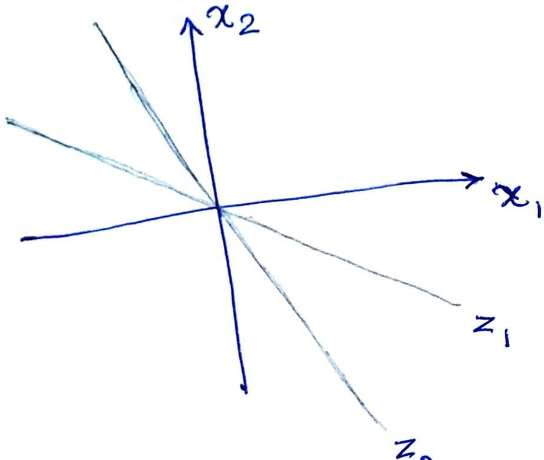
$X = MZ$, where M is a modal matrix,

the equation $\dot{X} = AX$ may be converted into the canonical form (for a 2nd order system)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where λ_1 and λ_2 are eigen values of matrix A, which are assumed to be distinct.

It should be noted that the transformation $X = MZ$ merely changes the coordinate system from (x_1, x_2) to (z_1, z_2) having the same origin. The new coordinate system (z_1, z_2) in general may not be rectangular.



We can write

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2 z_2$$

The trajectory traced out by the representative point P in the (z_1, z_2) plane has at any instant a resu velocity vector associated with the motion. The slope of the velocity vector is given by

velocity vector

$$\tan\theta = \frac{dz_2/dt}{dz_1/dt} = \frac{dz_2}{dz_1} = \frac{\dot{z}_2}{\dot{z}_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1}$$

$$\frac{dz_2}{dz_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1}$$

$$\Rightarrow \frac{dz_2}{z_2} = \left(\frac{\lambda_2}{\lambda_1}\right)\frac{dz_1}{z_1}$$

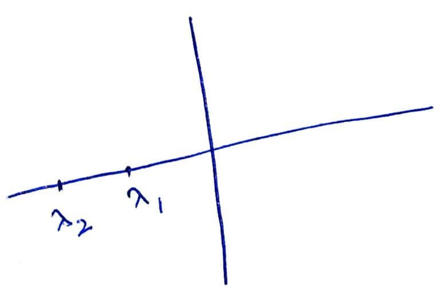
upon integration, we get,

$$z_2 = c\,(z_1)^{\lambda_2/\lambda_1},$$

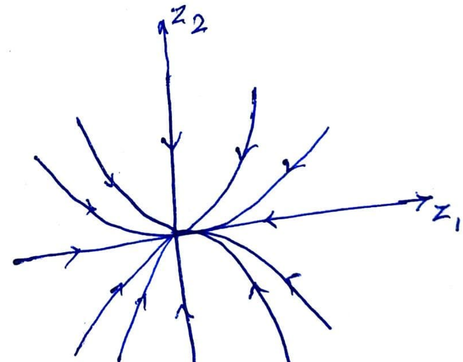
where c is the constant of integration

NODAL POINT:

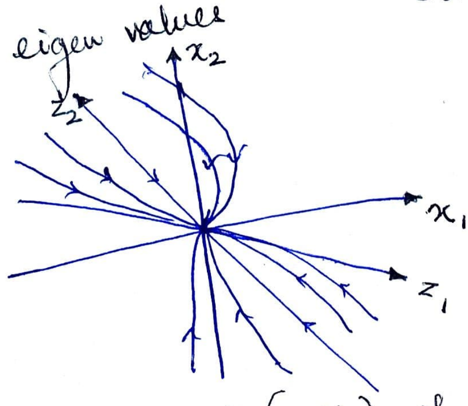
If eigen values (λ_1, λ_2) are both real and negative, all phase trajectories converge to origin, which is then called a nodal point. This is a stable nodal point. when (λ_1, λ_2) are both real and positive, the result is an unstable nodal point.



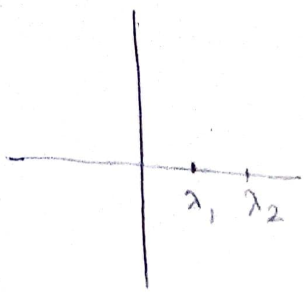
Location of eigen values

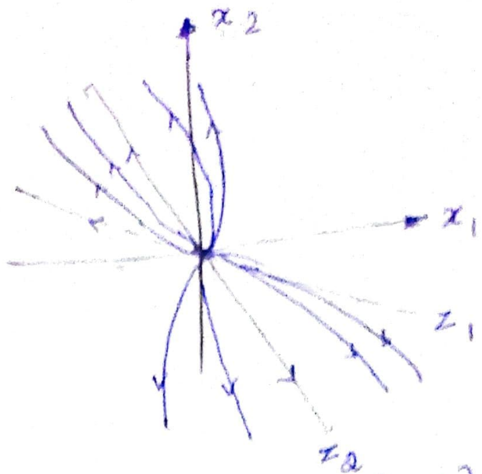


stable node in (z_1, z_2) plane.



stable node in (x_1, x_2) plane.



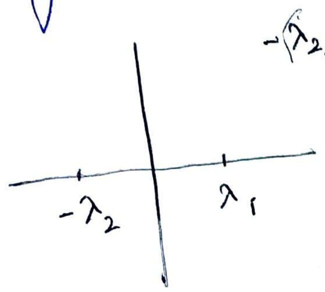


Unstable node in (x_1, x_2) plane

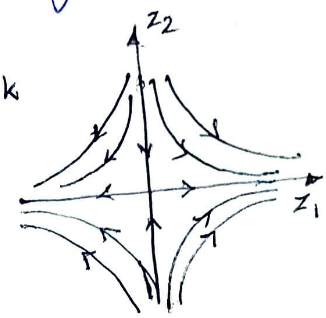
SADDLE POINT:

If both eigen values are real, equal and negative of each other, then the origin is a saddle point, which is always unstable.

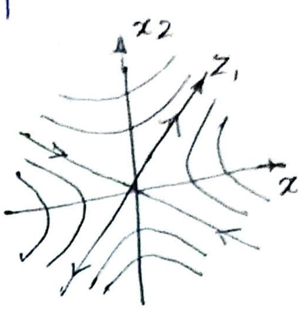
$-(\lambda_2/\lambda_1) = k$



Location of eigen values



Saddle Point in (z_1, z_2) plane

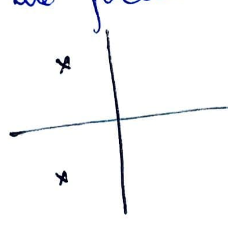


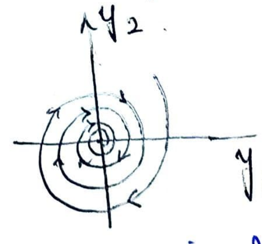
Saddle Point in (x_1, x_2) plane.

FOCUS POINT:

Let the eigen values be $\lambda_1, \lambda_2 = \sigma \pm j\omega$
= Complex conjugate pair

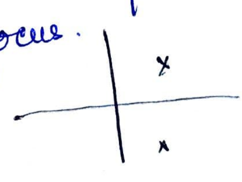
The phase portraits for negative real parts is shown below. The origin is a stable focus. The spiralling is towards the focus.

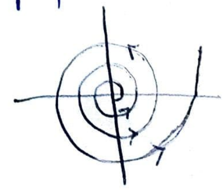




stable focus

If the eigen values are complex conjugate pairs with positive real parts, the spiralling is away from the origin, which is an unstable focus.





Unstable focus

CENTRE OR VORTEX POINT.

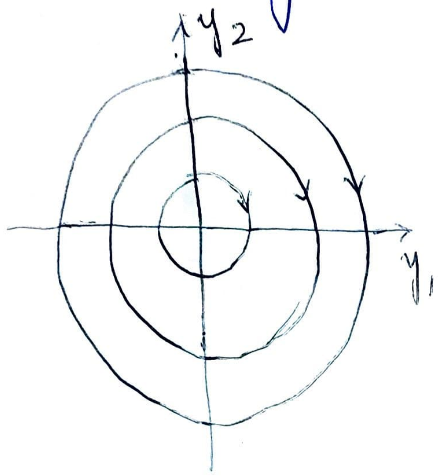
The eigen values are

$$\lambda_1, \lambda_2 = \pm j\omega$$

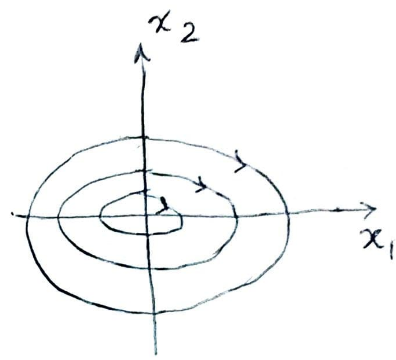
= On the imaginary axis.

The phase portrait has closed path trajectories and the origin is called a Centre.

The system is limitedly stable.



Centre in (y_1, y_2) plane



Centre in (x_1, x_2) plane.